

# Bubble dynamics: (nucleating) radiation inside dust

R. Casadio<sup>1,2,\*</sup> and A. Orlandi<sup>1,2,†</sup>

<sup>1</sup>*Dipartimento di Fisica, Università di Bologna, via Irnerio 46, 40126 Bologna, Italy*

<sup>2</sup>*INFN, Sezione di Bologna, Via Irnerio 46, I-40126 Bologna, Italy*

We consider two spatially flat Friedmann-Robertson-Walker spacetimes divided by a time-like thin shell in the nontrivial case in which the inner region of finite extension contains radiation and the outer region is filled with dust. We will then show that, while the evolution is determined by a large set of constraints, an analytical description for the evolution of the bubble radius can be obtained by formally expanding for short times after the shell attains its minimum size. In particular, we will find that a bubble of radiation, starting out with vanishing expansion speed, can be matched with an expanding dust exterior, but not with a collapsing dust exterior, regardless of the dust energy density. The former case can then be used to describe the nucleation of a bubble of radiation inside an expanding dust cloud, although the final configuration contains more energy than the initial dust, and the reverse process, with collapsing radiation transforming into collapsing dust, is therefore energetically favored. We however speculate a (small) decaying vacuum energy or cosmological constant inside dust could still trigger nucleation. Finally, our perturbative (yet analytical) approach can be easily adapted to different combinations of matter inside and outside the shell, as well as to more general surface density, of relevance for cosmology and studies of defect formation during phase transitions.

PACS numbers: 04.40.-b, 04.20.-q, 04.20.Cv, 98.80.Jk

## I. INTRODUCTION

The dynamics of an infinitely thin, spherically symmetric shell  $\Sigma$  separating two spacetime regions  $\Omega_{\pm}$  with given metrics is a well known problem of General Relativity. The general theory dates back to 1965 [1] and is completely understood. Given the symmetry of the system, we can use spherical coordinates  $x^{\mu}_{\pm} = \{t_{\pm}, r_{\pm}, \theta, \phi\}$  in  $\Omega_{\pm}$ , respectively, where the angular coordinates are the same in both patches, and  $0 \leq r_- < r_-^s$ ,  $r_+^s < r_+$ , with  $r_{\pm}^s = r_{\pm}^s(t_{\pm})$  the (in general time-dependent) radial coordinates of the shell in  $\Omega_{\pm}$ . One then takes specific solutions  $g_{\mu\nu}^{\pm}$  of the Einstein equations inside  $\Omega_{\pm}$  and imposes suitable junction conditions across  $\Sigma$ , namely the metric is required to be continuous across the shell,

$$g_{\mu\nu}^+|_{\Sigma} = g_{\mu\nu}^-|_{\Sigma} , \quad (1)$$

whereas the extrinsic curvature  $K_{ij}$  of  $\Sigma$  is allowed to have a jump proportional to the surface stress-energy tensor of the time-like shell  $\sigma_{ij}$  (italic indices run on the shell's three-dimensional world-sheet),

$$[K_i^j] - \delta_i^j [K_l^l] = \kappa \sigma_i^j , \quad (2)$$

in which  $[K_i^j] \equiv K_i^j|_+ - K_i^j|_-$  denotes the difference between extrinsic curvatures on the two sides of the shell. Note that we set  $c = 1$  and  $\kappa = 8\pi G_N/3 = \ell_P/M_P$  ( $= 1$  when convenient), where  $G_N$  is Newton's constant and  $\ell_P$  ( $M_P$ ) the Planck length (mass). Although the classical evolution equation (2) may appear simple, it has been

solved only in a few special cases, most notably for the vacuum or with cosmological constants in  $\Omega_{\pm}$  [2] (for an extensive bibliography see Ref. [3]).

A most intriguing result emerges in the semiclassical picture [4, 5], where one finds that “bubbles” can be quantum mechanically created from nothing (in a sense, at the expense of gravitational energy). This may occur when one has a classical solution for an expanding shell areal radius with a (finite) minimum value (turning point of the classical trajectory, larger than  $\ell_P$ ), and a non-vanishing quantum mechanical amplitude for the “tunneling” into such a system from one without the shell (that is, a shell of zero area). It has been conjectured that these bubbles could represent child universes generated inside a parent (or “landscape”) universe [6–8], if they expand indefinitely (or at least long enough). Bubble dynamics might also be used to model regions of space within which a matter phase transition occurs (from false to true vacuum, as well as between different form of matter [9]). One can, for instance, use such a model to approximate the formation of radiation from a decaying scalar field during reheating after inflation. It is known that for an inflaton with a quadratic potential, the time averaged dynamics of the final oscillation phase mimics that of matter. The approach developed in this paper could turn out to be suitable to describe the decay into radiation. In the end, knowing the correct evolution of such a bubble would be of great help in understanding how defects formed during phase transitions are “ironed out” by the expansion of the new phase.

In this paper, we are mainly interested in presenting an analytical (perturbative in time) approach to study a time-like shell's dynamics and to obtain analytical conditions for the existence of expanding bubbles in terms of the energy densities inside and outside the shell, when

\*Electronic address: [casadio@bo.infn.it](mailto:casadio@bo.infn.it)

†Electronic address: [orlandi@bo.infn.it](mailto:orlandi@bo.infn.it)

such regions contain homogeneous dust or radiation. This problem is made technically cumbersome because of the occurrence of algebraic constraints (to ensure the arguments of proliferating square roots are positive). We shall therefore consider in details only the specific case of nucleation of a spatially flat radiation bubble inside (spatially flat) collapsing or expanding dust, in order to keep the presentation of our method more streamlined. Nonetheless, these cases are also of particular physical interest. For example, one can conceive the density inside a collapsing astrophysical object might be large enough to allow for the creation of supersymmetric matter which, in turn, would then annihilate regular matter and produce a ball of radiation [10]. Likewise, in a dense matter-dominated expanding universe, one might consider the possibility of spontaneous nucleation of radiation bubbles. We shall find the dust energy density just sets the overall scale of the problem. Assuming the bubble surface density is (initially) constant, expanding radiation bubbles may then be matched with an expanding dust exterior, the time-like shell surface density being uniquely related to the inner radiation density. In order to view the bubble creation as a phase transition from dust to radiation (plus the surface density of the shell), one however needs an external source of energy, since the total energy of the bubble is larger than the initial energy of the dust. This implies that the reverse process of collapsing radiation turning into collapsing dust is actually favored energetically. Moreover, no configuration with expanding bubbles is allowed inside collapsing dust, regardless of how large is the dust energy density, and the conversion between the two types of matter therefore appears highly disfavored in this case.

In Section II, we briefly review the fundamental equations and constraints that describe general time-like bubble dynamics, following Ref. [9], and then specify all expressions for  $\Omega_{\pm}$  given by spatially flat Friedmann-Robertson-Walker (FRW) regions filled with dust or radiation. In Section III, we work out the explicit case of a radiation bubble of constant surface density nucleated inside collapsing or expanding dust, for which we obtain the initial minimum radius in terms of the inner and outer energy densities. Finally, in Section IV, we make some considerations about our findings and possible future generalizations.

## II. BUBBLE DYNAMICS

For our analysis, we will mostly follow the notation of Ref. [9], where the metric in each portion  $\Omega_{\pm}$  of spacetime is given by

$$ds^2 = e^{\nu(r,t)} dt^2 - e^{\lambda(r,t)} dr^2 - R^2(r,t) d\Omega^2, \quad (3)$$

where  $t = t_{\pm}$  are the time coordinates inside the corresponding patches, and likewise for the three spatial coordinates. On the shell time-like surface  $\Sigma$ , one has the

line element

$$ds^2|_{\Sigma} = d\tau^2 - \rho^2(\tau) d\Omega^2, \quad (4)$$

in which  $\tau$  is the proper time as measured by an observer at rest with the shell of areal radius  $\rho = R_{\pm}(r_{\pm}^s(t_{\pm}), t_{\pm})$ . The relation between  $\tau$  and  $t_{\pm}$  is obtained from the equation of continuity of the metric, Eq. (1), and is displayed below in Eq. (25) for the cases of interest. On solving Eq. (2) in terms of  $\rho$  and  $\dot{\rho} = d\rho/d\tau$ , one gets the dynamical equation

$$\dot{\rho}^2(\tau) = B^2(\tau) \rho^2(\tau) - 1, \quad (5)$$

where

$$B^2 = \frac{(\epsilon_+ + \epsilon_- + 9\kappa\sigma^2/4)^2 - 4\epsilon_- \epsilon_+}{9\sigma^2}, \quad (6)$$

with  $\sigma = \sigma_0^0(\tau)$  the shell's surface density and  $\epsilon_{\pm} = \epsilon_{\pm}(t_{\pm})$  the time-dependent energy densities in  $\Omega_{\pm}$  respectively. It is important to recall that metric junctions can involve different topologies for  $\Omega_{\pm}$ , but we are here considering only the so-called “black hole” type, in which both portions of spacetime have increasing area radii  $R_{\pm}$  in the outward direction (of increasing  $r_{\pm}$ ). Assuming the surface density of the shell is positive, one must then have<sup>1</sup>

$$\epsilon_+(\tau) - \epsilon_-(\tau) > \frac{9}{4}\kappa\sigma^2(\tau), \quad (7)$$

at all times, in order to preserve the chosen spacetime topology [9, 11].

In the pure vacuum case,  $\epsilon_{\pm}$  are constant and for constant  $\sigma$  the solution is straightforwardly given by

$$\rho(\tau) = B^{-1} \cosh(B\tau), \quad (8)$$

where  $B = B(\epsilon_{\pm}, \sigma)$  from Eq. (6) is also constant [2].

In the non-vacuum cases, finding a solution is however significantly more involved. Regardless of the matter content of  $\Omega_{\pm}$ , it is nonetheless possible to derive a few general results for a bubble which nucleates at a time  $\tau = \tau_0$ , that is a shell that expands from rest,

$$\dot{\rho}_0 = 0, \quad (9)$$

with initial finite (turning) radius ( $\rho_0 > 0$ ), where the subscript 0 will always indicate quantities evaluated at the time  $\tau = \tau_0$ . First of all, from Eq. (5), the initial radius must be given by

$$\rho_0 = |B_0^{-1}|, \quad (10)$$

which requires  $B_0$  real, or

$$(\epsilon_{0+} + \epsilon_{0-} + 9\kappa\sigma_0^2/4)^2 > 4\epsilon_{0-}\epsilon_{0+}. \quad (11)$$

---

<sup>1</sup> This also implies that  $\epsilon_{0+} > \epsilon_{0-}$  and, from the Friedmann equation (16) given below,  $H_+^2 > H_-^2$ .

This condition is always satisfied if  $\epsilon_0^{\text{dust}}$  and  $\epsilon_0^{\text{rad}}$  are both positive and will therefore be of no relevance in this paper, but must be carefully considered when allowing for negative energy densities (and non-vanishing spatial curvature). Further, upon deriving Eq. (5) with respect to  $\tau$  (always denoted by a dot)

$$2\dot{\rho}\ddot{\rho} = 2\left(B\dot{B}\rho^2 + B^2\rho\dot{\rho}\right), \quad (12)$$

and using Eq. (9), one also obtains

$$\dot{B}_0 = 0, \quad (13)$$

assuming  $\ddot{\rho}_0$  is not singular. The constraint in Eq. (7) at  $\tau = \tau_0$ ,

$$\epsilon_{0+} - \epsilon_{0-} > \frac{9}{4}\kappa\sigma_0^2, \quad (14)$$

and the conditions in Eqs. (9) and (13) will play a crucial role in the following.

### A. Flat FRW regions

Since we wish to study the particular case of a shell  $\Sigma$  separating two regions  $\Omega_{\pm}$  filled with homogeneous fluids, the metrics in  $\Omega_{\pm}$  will be taken to be the usual FRW expressions. Moreover, we already assumed  $\Omega_-$  has finite initial extension and, by definition, represents the interior of the shell. As a further simplification, we shall only consider flat spatial curvature and set the cosmological constant  $\Lambda = 0$  everywhere.

The metrics (3) in the inner and outer regions are therefore given by

$$ds^2 = dt^2 - a^2(t) [dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (15)$$

where  $a(t)$  is the scale factor which evolves according to the Friedmann equations

$$H^2 = \left(\frac{1}{a} \frac{da}{dt}\right)^2 = \kappa\epsilon \quad (16)$$

$$2\frac{1}{a} \frac{d^2a}{dt^2} + \left(\frac{1}{a} \frac{da}{dt}\right)^2 = -3\kappa p. \quad (17)$$

We assume the energy density  $\epsilon$  and pressure  $p$  of the fluids obey barotropic equations of state,

$$p = w\epsilon(t), \quad (18)$$

and recover the well-known behaviors

$$\epsilon(t) \left(\frac{a(t)}{a_0}\right)^{3(w+1)} = \epsilon_0, \quad (19)$$

in which  $\epsilon_0$  is the density evaluated at a reference instant of time  $t = t_0$  and  $a_0 = a(t_0)$ . For dust,  $w = 0$  ( $p = 0$ ),

whereas for radiation  $w = 1/3$ , so that

$$\begin{aligned} \epsilon^{\text{dust}}(t) &= \frac{\epsilon_0 a_0^3}{a^3(t)} \\ \epsilon^{\text{rad}}(t) &= \frac{\epsilon_0 a_0^4}{a^4(t)}. \end{aligned} \quad (20)$$

The evolution of scale factors in cosmic time for expanding ( $\uparrow$ ) and contracting ( $\downarrow$ ) solutions are finally given by

$$\begin{aligned} a_{\uparrow\downarrow}^{\text{rad}}(t) &= \left(\gamma \pm 2\sqrt{M^{\text{rad}}}t\right)^{1/2} \\ \frac{da_{\uparrow\downarrow}^{\text{rad}}}{dt} &= \pm \frac{\sqrt{M^{\text{rad}}}}{a^{\text{rad}}(t)} \end{aligned} \quad (21)$$

and

$$\begin{aligned} a_{\uparrow\downarrow}^{\text{dust}}(t) &= \left(\delta \pm \frac{3}{2}\sqrt{M^{\text{dust}}}t\right)^{2/3} \\ \frac{da_{\uparrow\downarrow}^{\text{dust}}}{dt} &= \pm \sqrt{\frac{M^{\text{dust}}}{a^{\text{dust}}(t)}}, \end{aligned} \quad (22)$$

where, in the above r.h.s., the  $+$  signs are for expansion and  $-$  signs for contraction,

$$\begin{aligned} M^{\text{rad}} &= \kappa a_0^4 \epsilon_0^{\text{rad}} \\ M^{\text{dust}} &= \kappa a_0^3 \epsilon_0^{\text{dust}}, \end{aligned} \quad (23)$$

and  $\gamma$  and  $\delta$  integration constants that determine the size of the scale factors at  $t = 0$ . Later, for convenience, we will set  $\gamma = \delta = 1$  at  $t = 0$ , so that  $\epsilon(0) = \epsilon_0$  and  $a(0) = a_0 = 1$ .

Let us now consider the time-like shell  $\Sigma$  at  $r_{\pm} = r_{\pm}^s(t_{\pm})$  separating the two regions  $\Omega_{\pm}$ . Clearly, metric continuity implies

$$\rho = a_{\pm}(t_{\pm}) r_{\pm}^s(t_{\pm}). \quad (24)$$

The inner and outer spaces are characterized by different physical parameters. In particular, as one can see from Eq. (6), the shell's dynamics are determined by:

1. The type of fluid inside the shell (its equation of state  $w_-$ );
2. The initial values  $a_{0-}$  and  $\epsilon_{0-}$ ;
3. The type of fluid outside the shell (its equation of state  $w_+$ );
4. The initial values  $a_{0+}$  and  $\epsilon_{0+}$ ;
5. The shell surface density  $\sigma$  (as a function of the radius  $\rho$ ).

A given configuration of dust, radiation and surface density is admissible only if the corresponding initial conditions are such that Eqs. (9), (13) and (14) are satisfied.

### B. Time transformations and expansion

The densities  $\epsilon_{\pm}$  in Eq. (20) are given in terms of coordinate times  $t_{\pm}$ . However, it is the time  $\tau$  measured by an observer on the shell which appears in the evolution equation (5). Hence we need to find the transformation from  $t_{\pm}$  to  $\tau$ . Following Ref [9], we recall that metric continuity (1) implies <sup>2</sup>

$$\left. \frac{dt_{\pm}}{d\tau} \right|_{\Sigma} = \left\{ \frac{H \rho \dot{\rho}}{\Delta} \left[ 1 \pm \sqrt{1 + \frac{\Delta^2 - \Delta(\dot{\rho}^2 + H^2 \rho^2)}{(H \rho \dot{\rho})^2}} \right] \right\}_{\pm}, \quad (25)$$

in which  $H$  is again the Hubble “constant”,  $\Delta = \kappa \epsilon \rho^2 - 1$ , and the expression within braces must be estimated on the two sides of the shell <sup>3</sup>. Now, the above two equations should be solved along with Eq. (5), which makes it clear why it is impossible to obtain general analytic solutions.

An important result can be obtained by considering the time when the bubble is at rest, that is  $t_{\pm} = t_{0\pm}$  and  $\tau = \tau_0$ , with  $\dot{\rho}(\tau_0) = 0$  and  $H_0 \equiv H(t_0)$ , namely

$$\left. \frac{dt_{\pm}}{d\tau} \right|_{\Sigma,0} = \pm \sqrt{1 - \frac{H_{0\pm}^2 \rho_0^2}{\Delta_{0\pm}}}. \quad (26)$$

From Eq. (16) we see that  $\Delta = H^2 \rho^2 - 1$ , therefore

$$\begin{aligned} \left. \frac{dt_{\pm}}{d\tau} \right|_{\Sigma,0} &= \frac{\pm 1}{\sqrt{1 - H_{0\pm}^2 \rho_0^2}} = \frac{\pm 1}{\sqrt{1 - \kappa \epsilon_{0\pm} \rho_0^2}} \\ &= \frac{\pm 1}{\sqrt{-\Delta_{0\pm}}}, \end{aligned} \quad (27)$$

with the signs in the numerator simply reflecting the directions  $t_{\pm}$  flow relative to  $\tau$ . It is clear that real solutions to Eq. (27) exist only if

$$\Delta_{0\pm} < 0. \quad (28)$$

Remarkably, this is the same as stating that the energy density inside the radius  $\rho = \rho_0$  must not generate a black hole, as one can easily check by considering the Schwarzschild radius  $r_S = 2 G_N M$  with  $M = (4\pi/3) \epsilon_0 \rho_0^3$ . This is manifest when considering  $\epsilon_{0-}$  inside the bubble, but it must also hold for the energy density  $\epsilon_{0+}$  outside the bubble. For the outer region, this means the bubble must lie inside the Hubble radius,  $\rho_0 < H_0^{-1}$ . Putting together these conditions tells us that the temporal coordinates are properly transformed only within a causal region of the spacetime.

In order to study how the bubble grows after nucleation, we can expand  $t = t(\tau)$  for short times about  $t_{0\pm}$

and  $\tau_0$ . Further, we want all times directed the same way, so we choose the + sign in the above expression and obtain, to linear order,

$$t_{\pm} \simeq t_{0\pm} + \frac{\tau - \tau_0}{\sqrt{-\Delta_{0\pm}}}, \quad (29)$$

where  $t_{0\pm}$  are integration constants.

Unfortunately, a first order expansion is not sufficient to study the evolution of the bubble radius. Since  $\dot{\rho}_0 = 0$ , we need at least second order terms in  $\tau$  to get significant results, which makes all expressions very cumbersome. We shall therefore just consider a few specific cases, generalizing the exact result (8) for a shell of constant surface energy in vacuum. For such cases, our perturbative approach will yield exact conditions for the bubble’s existence, which we see as a clear advantage with respect to purely numerical solutions. Other advantages would be that having analytical expressions is a necessary ingredient for quantum mechanical (or semiclassical) studies of these systems. Moreover, adapting our procedure to all possible combinations of fluids in  $\Omega_{\pm}$ , and for more general shell surface density, should be rather straightforward.

### III. RADIATION BUBBLE INSIDE DUST

The main idea in our approach stems from the observation that the (three) fundamental (first order differential) equations (5) and (25) contain six functions of the proper time  $\tau$ : the shell radius  $\rho$ , its surface density  $\sigma$ , the two times  $t_{\pm}$  and the two Hubble functions  $H_{\pm}$ . Once we choose the matter content inside  $\Omega_{\pm}$  and on the shell, the Hubble functions and surface density are uniquely fixed and we are left with the three unknowns  $\rho$  and  $t_{\pm}$  (and a set of constraints for the initial conditions). To determine these unknowns, we find it convenient to formally expand the shell radius  $\rho$  and Hubble functions  $H_{\pm}$  for short (proper) time “after the nucleation of the bubble” (when  $\dot{\rho}_0 = 0$ ), and solve Eqs. (5) and (25) order by order.

Since expressions rapidly become involved, and a general treatment for all combinations of matter content in  $\Omega_{\pm}$  and shell surface density would be hardly readable, we shall only consider the specific case of a radiation bubble ( $w_- = 1/3$ ) inside a region filled with dust ( $w_+ = 0$ ). We further assume the shell’s surface density

$$\sigma(\tau) = \sigma_0 > 0 \quad (30)$$

is constant and positive. Since one would expect the shell’s density decreases as the shell’s surface grows, this assumption might appear rather strong. However, it is the simplest way to ensure the junction remains of the “black hole” type, and a more thorough discussion of this point can be found in Ref. [9]. In order to keep the presentation uncluttered, we also set  $\kappa = 1$  from

<sup>2</sup> There is a typo in Eq. (B10) of Ref [9]:  $\rho^2$  is missing in the last term in the square root.

<sup>3</sup> Note the sign ambiguity  $\pm$  in front of the square root just reflects the double root of a second degree equation and is *not* associated with the interior or exterior regions.

now on and regard all quantities as dimensionless (tantamount to assuming they are rescaled by suitable powers of  $\kappa = \ell_P/M_P$ ). This means that densities will be measured in Planck units, that is  $\epsilon = 1$  corresponds to the Planck density  $\epsilon_P = M_P/\ell_P^3 = \ell_P^{-2}$  and  $\sigma = 1$  to  $\sigma_P = M_P/\ell_P^2 = \ell_P^{-1}$ . Likewise,  $\rho = 1$  is the Planck length  $\ell_P$ . We also express the shell surface density and radiation energy density as fractions of  $\epsilon_0^{\text{dust}} > 0$ ,

$$\epsilon_0^{\text{rad}} = \epsilon_0^{\text{dust}} x, \quad \sigma_0 = \sqrt{\epsilon_0^{\text{dust}}} y, \quad (31)$$

with  $0 \leq x \leq 1$  and  $y \geq 0$ .

It is natural to choose  $\tau_0 = 0$ , and then proceed to analyze Eqs. (5) and (25) by formally expanding all relevant time-dependent functions in powers of  $\tau - \tau_0 = \tau$ :

**Step 1)** since  $\dot{\rho}_0 = 0$  [see Eq. (9)], we can formally write the bubble radius as

$$\rho(\tau) = \rho_0 + \frac{1}{2} \ddot{\rho}_0 \tau^2 + \mathcal{O}(\tau^3), \quad (32)$$

where  $\rho_0$  and  $\ddot{\rho}_0$  are parameters to be determined. In particular, from Eqs. (6) and (10), we obtain the (not yet final) expression

$$\begin{aligned} \rho_0 &= \frac{3\sigma_0}{\sqrt{(\epsilon_{0+} + \epsilon_{0-} + 9\kappa\sigma_0^2/4)^2 - 4\epsilon_{0-}\epsilon_{0+}}} \\ &= \frac{3(\epsilon_0^{\text{dust}})^{-1/2} y}{\sqrt{(1+x+9y^2/4)^2 - 4x}}, \end{aligned} \quad (33)$$

which only depends on  $\epsilon_{0\pm}$  and  $\sigma_0$ . More precisely,  $\epsilon_0^{\text{dust}}$  sets the overall scale of the shell radius and the fractions  $x$  and  $y$  defined in Eq. (31) the detailed form.

We next obtain  $t_{\pm} = t_{\pm}(\tau)$  by solving Eq. (25). However, for this purpose we need the Hubble parameters  $H_{\pm}$  as functions of  $\tau$ , whereas they explicitly depend on  $t_{\pm}$ :

**Step 2)** we replace  $H$  in Eq. (25) with the formal expansion

$$\mathcal{H} = \mathcal{H}_0 + \dot{\mathcal{H}}_0 \tau + \mathcal{O}(\tau^2), \quad (34)$$

where  $\mathcal{H}_0$  and  $\dot{\mathcal{H}}_0$  are unknown constant quantities to be determined by consistency. By expanding the right hand side of Eq. (25) to first order in  $\tau$  and then integrating, we obtain  $t$  to second order in  $\tau$ ,

$$t_{\pm} \simeq t_{0\pm} + \frac{\tau}{\sqrt{-\Delta_{0\pm}}} + \frac{\rho_0 \mathcal{H}_{0\pm}}{2\Delta_{0\pm}} \left( \ddot{\rho}_0 - \frac{\rho_0 \dot{\mathcal{H}}_{0\pm}}{\sqrt{-\Delta_{0\pm}}} \right) \tau^2, \quad (35)$$

where  $\rho_0$  must now be understood as the expression given in Eq. (33) and  $t_{0\pm}$  are integration constants we can set to zero without loss of generality. In fact, let us assume we are at rest in an expanding (or contracting) universe, corresponding to the old exterior phase (with parameters  $\epsilon_{0+}$  and  $H_{0+}$ ), and measure a time  $t'_+$  from the “Big Bang” (or beginning of collapse) of this exterior universe [ $a_+(t'_+ = 0) = 0$  or  $a_+(t'_+ = 0) > a(t'_+)$  for  $t'_+ > 0$ ,

respectively]. If, for instance, a new phase bubble arises at rest at the instant  $t'_+ = t'_{0+}$ , we can define  $t_+ = t'_+ - t'_{0+}$ , so that the bubble is created at  $t_+ = 0$ , and also set  $t_{0-} = 0$ , since an “inner time” is meaningless before any “inner part” exists. For different pictures, similar arguments can likewise be formulated.

**Step 3)** From Eq. (21) and (22), we choose an expanding radiation interior and contracting or expanding dust exterior,

$$a_- = a_{\uparrow}^{\text{rad}}(t_-), \quad a_+ = a_{\downarrow}^{\text{dust}}(t_+), \quad (36)$$

set  $a_{0\pm} = 1$  and express  $t_{\pm}$  according to Eq. (35). In so doing,  $a_{\pm}$  and  $da_{\pm}/dt_{\pm}$  become explicit functions of  $\tau$  containing  $\rho_0$ ,  $\mathcal{H}_{0\pm}$  and  $\dot{\mathcal{H}}_{0\pm}$ . For consistency with Eq. (34), we must therefore require

$$H(\tau) = \frac{1}{a} \frac{da}{d\tau} = \mathcal{H}(\tau), \quad (37)$$

with  $H_- = \mathcal{H}_- > 0$  and  $H_+ = \mathcal{H}_+ < 0$  for collapsing dust, or  $H_+ = \mathcal{H}_+ > 0$  for expanding dust.

**Step 4)** To zero order in  $\tau$ , Eq. (37) gives rise to a first-order equation for  $\mathcal{H}_0$ ,

$$H_0 = \frac{1}{a_0} \frac{da}{dt} \Big|_{\tau=0} = \frac{da}{dt} \Big|_{\tau=0} = \mathcal{H}_0. \quad (38)$$

The solutions are uniquely given by

$$\mathcal{H}_0^{\uparrow\downarrow} = \pm \sqrt{\epsilon_0}, \quad (39)$$

in which the  $\uparrow$  and  $+$  sign (respectively  $\downarrow$  and  $-$  sign) refer to expanding (contracting) solutions, i.e. solutions with increasing (decreasing) scale factor, as before<sup>4</sup>. Note this result also follows directly from the Friedmann equation (16) for  $\tau = t = 0$ .

To first order in  $\tau$ , one analogously obtains

$$\dot{\mathcal{H}}_0^{\uparrow\downarrow} = -\frac{n\epsilon_0}{\sqrt{1-\rho_0^2\epsilon_0}}, \quad (40)$$

with  $n = 2$  for radiation and  $n = 3/2$  for dust, and  $\rho_0$  must again be understood as the expression given in Eq. (33).

**Step 5)** Replace  $\rho_0$  from Eq. (33) and the chosen combination of Hubble parameters (39) inside  $\dot{B}_0$ , which must then satisfy Eq. (13). This equation will only contain  $\epsilon_{0-} = \epsilon_0^{\text{rad}}$ ,  $\epsilon_{0+} = \epsilon_0^{\text{dust}}$  and  $\sigma_0$ , so that it can be used to determine  $\sigma_0 = \sigma_0(\epsilon_{0-}, \epsilon_{0+})$ . In particular, introducing the fractions in Eq. (31), we obtain

$$\dot{B}_0 = \epsilon_0^{\text{dust}} \dot{b}_0(x, y), \quad (41)$$

<sup>4</sup> Note that, for example, the Hubble parameter for the expanding interior phase will carry a second subscript sign and will then be denoted as  $\mathcal{H}_{-}^{\uparrow}$ , where the subscript  $-$  indicates the inner region and the apex  $\uparrow$  stands for expansion.

from which it appears that the dust energy density just sets the overall scale like in Eq. (33). For any given values of  $\epsilon_0^{\text{dust}}$ , the shell surface density is instead determined by the radiation energy density according to

$$\dot{b}_0(x, y) = 0, \quad (42)$$

which, for the cases of interest, is a fourth-order algebraic equation for  $y$ . Analytic solutions can be found (in suitable ranges of  $x$ ), which we denote as  $\bar{y} = \bar{y}(x)$ , so that the allowed surface densities are given by

$$\bar{\sigma}_0 = \sqrt{\epsilon_0^{\text{dust}}} \bar{y}. \quad (43)$$

**Step 6)** Replace the above surface density  $\bar{\sigma}_0$  into the initial radius (33) and obtain its final expression,

$$\bar{\rho}_0 = \frac{3(\epsilon_0^{\text{dust}})^{-1/2} \bar{y}}{\sqrt{(1+x+9\bar{y}^2/4)^2 - 4x}}, \quad (44)$$

which can then be used to determine the final forms of  $\dot{\mathcal{H}}_{0\pm}$  and the scale factors  $a_{\pm}$  to first order in  $\tau$ .

**Step 7)** One must now check that  $\bar{\sigma}_0$  and  $\bar{\rho}_0$  satisfy all of the initial constraints and lead to valid time transformations (25), at least for some values of  $\epsilon_0^{\text{rad}}$  and  $\epsilon_0^{\text{dust}}$ . If not all of these conditions are met, one must conclude the corresponding physical system may not exist. Moreover, we note the condition (28) requires  $\epsilon_0 \lesssim \epsilon_P$  in order to have a (semi)classical bubble with  $\rho_0 \gtrsim \ell_P$ . In the following Section, one should therefore consider only dust and radiation energy densities  $\epsilon_0 \ll \epsilon_P$  and look at the limiting case  $\epsilon_0 \simeq \epsilon_P$  as a glimpse into the quantum gravity regime.

If a consistent solution for  $\bar{\sigma}_0$  and  $\bar{\rho}_0$  exists, one can proceed to determine higher orders terms (in  $\tau$ ). However, due to the increasing degree of complexity of the resulting expressions, we shall not go any further here. We instead present our findings for the two cases of interest separately.

### A. Collapsing dust

This case is defined by choosing the scale factors

$$a_- = a_{\uparrow}^{\text{rad}}(t_-), \quad a_+ = a_{\downarrow}^{\text{dust}}(t_+), \quad (45)$$

and proceeding as described above. We can then prove that this case does not admit solutions, in general, by simply analyzing the constraint (42),

$$\begin{aligned} \dot{b}_0 \propto 16x^2(3+4x^{3/2}) + 4x^{3/2}(4-9y^2)(4-8x-9y^2) \\ + 3(4+9y^2)(4-8x+9y^2) = 0, \end{aligned} \quad (46)$$

which admits the four solutions

$$\bar{y}_{\pm\pm} = \pm \frac{2}{3} \sqrt{(1-x) \frac{2x^{3/4} \pm i\sqrt{3}}{2x^{3/4} \mp i\sqrt{3}}}. \quad (47)$$

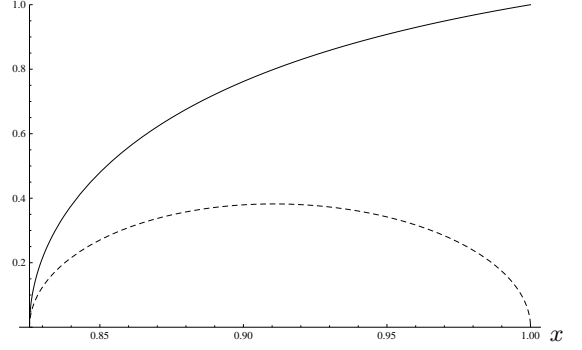


FIG. 1: Plot of  $\bar{\sigma}_0/\sqrt{\epsilon_0^{\text{dust}}} = \bar{y}_{+-}(x)$  (magnified by a factor of 10 for convenience, dashed line) and corresponding  $\bar{\rho}_0/(\epsilon_0^{\text{dust}})^{-1/2}$  (solid line) in the range (53).

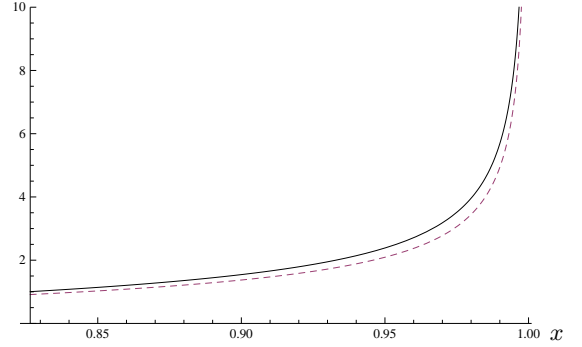


FIG. 2: Plot of  $\dot{t}_{0+}$  (solid line) and  $\dot{t}_{0-}$  (dashed line) for  $y = \bar{y}_{+-}(x)$  in the range (53).

However, for  $x \neq 1$ , all the  $\bar{y}_{\pm\pm}$  are complex and a complex surface density is obviously unphysical. One is then apparently left with the only trivial case  $x = 1$ , corresponding to  $\bar{y} = 0$  and

$$\bar{\rho}_0 = (\epsilon_0^{\text{dust}})^{-1/2} \sqrt{\frac{3+4x^{3/2}}{3+4x^{1/2}}} = 1. \quad (48)$$

This case however does not satisfy all the required constraints. For example, Eq. (14) for  $\sigma_0 = 0$  yields

$$\epsilon_0^{\text{dust}} > \epsilon_0^{\text{rad}} = x \epsilon_0^{\text{dust}}, \quad (49)$$

which clearly contradicts  $x = 1$ . Correspondingly, the time transformations (25) are not well-defined, because  $\dot{t}_{0+} = \dot{t}_{+}(\tau = 0; x)$  is complex for  $0 < x < 1$  and both  $\dot{t}_{0\pm}$  diverge for  $x \rightarrow 1$ .

The overall conclusion is thus that expanding radiation bubbles with a turning point of minimum radius cannot be matched with a collapsing dust exterior.

### B. Expanding dust

This case is defined by choosing the scale factors

$$a_- = a_{\uparrow}^{\text{rad}}(t_-), \quad a_+ = a_{\uparrow}^{\text{dust}}(t_+). \quad (50)$$

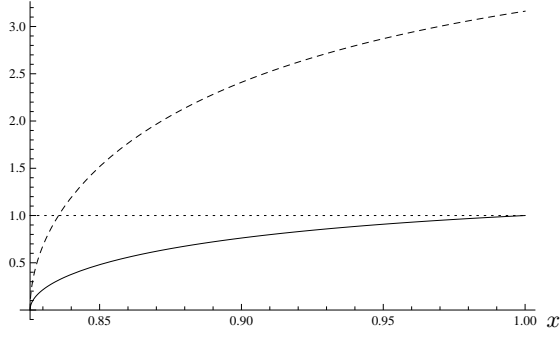


FIG. 3: Plot of  $\bar{\rho}_0$  for  $y = \bar{y}_{+-}(x)$  with  $\epsilon_0^{\text{dust}} = \epsilon_P/10$  (dashed line) and  $\epsilon_0^{\text{dust}} = \epsilon_P$  (solid line) in the range (53). Only values above  $\ell_P = 1$  represent acceptable semiclassical radii.

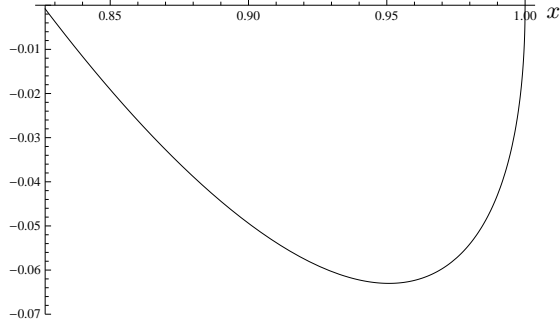


FIG. 4: Plot of  $\bar{C}_0/\bar{M}_0^{\text{dust}}$  for  $\bar{\sigma}_0 > 0$  and  $y = \bar{y}_{+-}(x)$  in the range (53).

The crucial task is again to solve the constraint in Eq. (42), namely

$$\dot{b}_0 \propto 16x^2(3 - 4x^{3/2}) - 4x^{3/2}(4 - 9y^2)(4 - 8x - 9y^2) - 3(4 + 9y^2)(4 - 8x + 9y^2) = 0, \quad (51)$$

admitting the four solutions

$$\bar{y}_{\pm\pm} = \pm \frac{2}{3} \sqrt{(1-x) \frac{2x^{3/4} \pm \sqrt{3}}{2x^{3/4} \mp \sqrt{3}}}, \quad (52)$$

which are real for

$$x_{\min} = \left(\frac{3}{4}\right)^{2/3} < x < 1. \quad (53)$$

We discard the negative solutions  $\bar{y}_{-\pm}$  associated to negative surface densities and just analyze the positive cases  $\bar{y}_{++}$ . It is then easy to see that  $\bar{y}_{++}$  leads to a surface density that diverges for  $x \rightarrow x_{\min}$ , and is always too large to satisfy the condition (14), since

$$1 - x - \frac{9}{4} \bar{y}_{++}^2 = \frac{\sqrt{3}(x-1)}{2x^{3/4} - \sqrt{3}} < 0, \quad (54)$$

in the range (53). In the limit for  $x \rightarrow 1$ ,  $\bar{y}_{++} \rightarrow 0$ , however, Eq. (14) is still violated in the strict sense and one can in fact show that  $\dot{t}_{0+}$  diverges.

The only solution which appears consistent is therefore

$$\begin{aligned} \bar{\sigma}_0 &= \sqrt{\epsilon_0^{\text{dust}}} \bar{y}_{+-} \\ &= \frac{2}{3} \sqrt{\epsilon_0^{\text{dust}}} \sqrt{(1-x) \frac{2x^{3/4} - \sqrt{3}}{2x^{3/4} + \sqrt{3}}}, \end{aligned} \quad (55)$$

with  $x$  again in the range (53). This expression yields a vanishing surface density for the limiting values  $x \rightarrow 1$  and  $x \rightarrow x_{\min}$  (see Fig. 1) and further satisfies the condition (14),

$$1 - x - \frac{9}{4} \bar{y}_{+-}^2 = \frac{\sqrt{3}(1-x)}{2x^{3/4} + \sqrt{3}} > 0. \quad (56)$$

The corresponding initial bubble radius is an increasing function of  $x$  (see Fig.1),

$$\bar{\rho}_0 = (\epsilon_0^{\text{dust}})^{-1/2} \sqrt{\frac{4x^{3/2} - 3}{x(4x^{1/2} - 3)}} < (\epsilon_0^{\text{dust}})^{-1/2}, \quad (57)$$

with  $\bar{\rho}_0(x \rightarrow 1) = (\epsilon_0^{\text{dust}})^{-1/2}$ . Further, the products

$$\epsilon_0^{\text{dust}} \bar{\rho}_0^2 < 1 \quad \text{and} \quad \epsilon_0^{\text{rad}} \bar{\rho}_0^2 < 1, \quad (58)$$

for  $x_{\min} < x < 1$ , as required by the condition (28). In fact the initial time derivatives  $\dot{t}_{0\pm}$  are well defined in this range (see Fig. 2) and only diverge for  $x \rightarrow 1$ . Note the above initial radius can be larger than  $\ell_P$  only if  $\epsilon_0^{\text{dust}} < \epsilon_P$  and for sufficiently large  $x$ , since  $\bar{\rho}_0 \rightarrow 0$  for  $x \rightarrow x_{\min}$  (see, for example, Fig. 3).

Finally, let us check if one can use the process of bubble nucleation to describe a phase transition from dust to radiation for the matter inside the sphere of radius  $\bar{\rho}_0$ , accompanied by the creation of a layer of non-vanishing surface density  $\bar{\sigma}_0$ . From Eqs. (55) and (57), one has

$$\bar{C}_0 \equiv \bar{M}_0^{\text{dust}} - \bar{M}_0^{\text{rad}} - \bar{E}_0^\Sigma < 0, \quad (59)$$

which means the dust energy inside the sphere of radius  $\bar{\rho}_0$  at time of bubble formation,  $\bar{M}_0^{\text{dust}} = (4\pi/3) \bar{\rho}_0^3 \epsilon_0^{\text{dust}}$ , is not sufficient to produce the radiation energy  $\bar{M}_0^{\text{rad}} = (4\pi/3) \bar{\rho}_0^3 \epsilon_0^{\text{rad}}$  and surface energy  $\bar{E}_0^\Sigma = 4\pi \bar{\sigma}_0 \bar{\rho}_0^2$ . An extra source is thus needed to provide the energy  $-\bar{C}_0 > 0$ . The reverse process of collapsing radiation reaching a minimum size  $\rho = \bar{\rho}_0$  and then turning into collapsing dust would instead be energetically favored, with the amount of energy  $-\bar{C}_0$  now being released. Of course, in order to support this kind of argument, the extra contribution should be a small perturbation on the given background,

$$|\bar{C}_0| \ll \bar{M}_0^{\text{dust}}, \quad (60)$$

since it was not included in the dynamical equations. From Fig. (4), we expect this is indeed a very good approximation since  $0 < -\bar{C}_0 \lesssim 0.06 \bar{M}_0^{\text{dust}}$ .

#### IV. CONCLUSIONS AND OUTLOOK

In this paper, we have analyzed bubbles of radiation whose time-like surface starts to expand inside collapsing or expanding dust with vanishing initial rate, and with the further (simplifying) assumptions that the bubble's surface density is constant and positive, and both interior and exterior are spatially flat. These bubbles generalize the simplest self-gravitating case of a shell with constant surface density expanding in vacuum, for which the exact trajectories are known [1, 3]. These generalizations are of potential interest both for the physics of the early universe and the description of astrophysical processes. However, although the general formalism was already developed a long time ago [1], and the dynamics are ruled by apparently simple equations [9], finding explicit solutions is not straightforward.

By developing an approach to obtain analytical expressions for the evolution of the bubble radius in the shell's proper time,  $\rho = \rho(\tau)$  with  $\dot{\rho}(\tau = 0) = 0$ , we determined the conditions which allow for the existence of such configurations. Although our approach is perturbative (with an expansion for short times after nucleation), the conditions for the bubble's existence are exact, which is a clear advantage with respect to purely numerical solutions. We then found that expanding radiation bubbles of constant surface density may not be matched to a collapsing dust exterior. More precisely, we found that inside collapsing dust there may not be a bubble of radiation whose surface ever reaches vanishing speed of expansion at finite radius. Bubbles whose radius admits a turning point are instead allowed inside an expanding dust-dominated universe. They further can be used to model a phase transition from radiation to dust if an external source provides part of the the energy required to build the shell, or the converse process with release of energy (albeit, of an amount small enough to leave the background configuration unaffected).

Let us clarify this point about energy conservation. The fundamental Eqs. (5) and (25) are just a different form of the junction equations (1) and (2) which, in turn, follow from the Einstein equations. Conservation of the energy-momentum in a given system is therefore guaranteed. However, when we use bubbles to model a phase transition, we are considering the possibility that a region of space filled with dust be replaced by radiation enclosed inside an expanding shell, or the reverse process. Technically, we are therefore considering two different systems: one with dust and one with a bubble of radiation within a shell of positive surface density whose radius evolves along a trajectory with a turning point (zero speed at finite minimum radius). The total energy in the two configurations differ by the amount  $\bar{C}_0$  defined in Eq. (59), and a (quantum) transition between them would there-

fore violate energy conservation and be suppressed in the semiclassical regime. By looking at Fig. 4, we however see that  $|\bar{C}_0| \ll \bar{M}_0^{\text{dust}}$ . One may thus argue the unspecified matter contribution carrying the energy  $|\bar{C}_0|$  should be well approximated as a perturbation with respect to the dust and radiation, with its backreaction on the chosen configuration consistently negligible. If so, one can further speculate if the extra energy required to nucleate radiation could be provided by pressure in the initial cloud or by the decay of a region of false vacuum (with vacuum energy or cosmological constant  $\Lambda_+$ ) to true vacuum (with cosmological constant  $\Lambda_- < \Lambda_+$ ), like in the seminal Refs. [4].

The fact that no consistent solution was found inside collapsing dust does not mean that expanding radiation bubbles may not be produced at all in this context, which includes, for example, the collapsing core of a supernova or other astrophysical processes leading to black hole formation. In fact, the situation might change if one, for instance (and more realistically), includes matter pressure or a radius-dependent surface density,  $\sigma = \sigma(\rho)$ . This observation thus brings us to briefly comment on the possible generalizations and extensions of the present work, which include the just mentioned non-constant  $\sigma$ , as well as different combinations of matter inside and outside the shell, and cosmological constant(s)  $\Lambda_{\pm}$ . Moreover, one might like to consider the vacuum inside the shell and radiation outside (with or without  $\Lambda_{\pm}$ ) and apply the corresponding results to the thick shell model previously studied in Refs. [12].

Finally, our analysis is entirely based on classical General Relativity and no attempt was made to compute the quantum mechanical “tunneling” probability for radiation bubbles to come into existence (or convert to dust). Such an analysis requires the (effective Euclidean) action to be integrated along the (classically forbidden) trajectory for the bubble radius  $\rho$  to go from 0 to  $\rho_0$  [5], whose construction is clearly no easy task, given the classical trajectories are so difficult to determine. Nonetheless, another advantage of our approach is that it provides analytical (albeit perturbative) expressions, which is a property one needs for any quantum mechanical (or semiclassical) studies of these systems. Of course, energy densities above the Planck scale would not be meaningful in this context, since one then has no guarantee the dynamical equations derived from General Relativity can be trusted at all.

#### Acknowledgments

We would like to thank G.L. Alberghi and S. Ansoldi for many discussions about the topic. R.C. and A.O. are supported by INFN grant BO11.

---

[1] W. Israel, *Nuovo Cimento* **44B**, 1 (1966); [Erratum: **48B**, 463 (1967)].

[2] S.K. Blau, E.I. Guendelman and A.H. Guth, *Phys. Rev.*



- D **35**, 1747 (1987).
- [3] S. Ansoldi, Class. Quant. Grav. **19**, 6321 (2002).
  - [4] S.R. Coleman, Phys. Rev. D **15**, 2929 (1977) [Erratum-ibid. D **16**, 1248 (1977)]; S.R. Coleman and F. De Luccia, Phys. Rev. D **21**, 3305 (1980).
  - [5] E. Farhi, A.H. Guth and J. Guven, Nucl. Phys. B **339**, 417 (1990); S. Ansoldi, A. Aurilia, R. Balbinot and E. Spallucci, Class. Quant. Grav. **14**, 2727 (1997).
  - [6] Y.S. Piao, Nucl. Phys. B **803**, 194 (2008).
  - [7] A.H. Guth and E.J. Weinberg, Nucl. Phys. B **212**, 321 (1983).
  - [8] D. Yamauchi, A. Linde, A. Naruko, M. Sasaki and T. Tanaka, “Open inflation in the landscape,” arXiv:1105.2674 [in].
  - [9] V.A. Berezin, V.A. Kuzmin and I.I. Tkachev, Phys. Rev. D **36**, 2919 (1987).
  - [10] L. Clavelli, High Energy Dens. Phys. **0606**, 002 (2006); [High Energy Dens. Phys. **2**, 97 (2006)]; P.L. Biermann and L. Clavelli, “A Supersymmetric model for triggering Supernova Ia in isolated white dwarfs,” arXiv:1011.1687.
  - [11] N. Sakai and K.i. Maeda, Prog. Theor. Phys. **90**, 1001 (1993); N. Sakai and K.i. Maeda, Phys. Rev. D **50**, 5425 (1994).
  - [12] G.L. Alberghi, R. Casadio, G.P. Vacca and G. Venturi, Class. Quant. Grav. **16**, 131 (1999); Phys. Rev. D **64**, 104012 (2001); G.L. Alberghi, R. Casadio and G. Venturi, Phys. Rev. D **60**, 124018 (1999); G.L. Alberghi, R. Casadio and D. Fazi, Class. Quant. Grav. **23**, 1493 (2006).